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ON THE LATERAL STABILITY OF MULTI-STORY BENTS

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ON THE LATERAL STABILITY OF MULTI-STORY BENTS[1]

E. F. Masur[2]

SUMMARY

A numerical method is described which makes it possible to determine the stability of multi-story bents subject to sidesway without the solution of the classical transcendental equations. Through the use of moment distribution techniques, which are familiar in conventional structural design problems, the desired safety factor is boxed in between upper and lower bounds. Through a repetition of the procedure, the gap can be narrowed down to within any required degree of accuracy.

INTRODUCTION

The problem of the stability of multi-story bents against buckling in the form of sidesway of the floors has been treated thoroughly during the past two decades. Starting with the work of F. Bleich [1][3] further investigations were carried out by Chwalla [2] and others and are comprehensively treated by F. Bleich and H. Bleich in [3]. Basically, the solution of the problem as developed in the above mentioned references consists of setting up a number of linear equations which express either the conditions of equilibrium in terms of the displacements or the equations of compatible deformations in terms of the moments and shears. In either case, a system of linear homogeneous equations is arrived at which admits a non-trivial solution if, and only if, the determinant of the coefficients vanishes. In this fashion the "characteristic equation" governing the state of neutral equilibrium is developed.

While the establishment of this set of equations is reasonably systematic, the solution of the determinantal equation presents formidable numerical difficulties as far as standard engineering office design methods are concerned. A measure of simplification can be introduced if the structure under consideration exhibits a sufficient degree of regularity to enable the investigator to make certain simplifying assumptions as to the nature of the expected fundamental mode of buckling. The problem can then be solved readily, though only approximately, through the use of energy methods or through the establishment of a difference equation. For the general type of bent, however, an easily computable exact solution is practically out of reach.

It is the purpose of this paper to present an alternate approach to the problem at hand. This approach avoids the solution of a transcendental equation; instead it employs only algebraic linear equations which are set up through the use of moment distribution techniques which are now common in

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3. Numbers in brackets refer to the bibliography at the end of this paper.

the structural engineering field. By means of a theorem developed in the paper, lower and upper bounds to the critical loads are computed. The gap between these bounds may be narrowed down arbitrarily.

"A similar technique was first suggested by Winter and his associates [8] in the solution of the stability problem of a one-story bent. Their approach was based on the physically apparent fact that a frame is stable so long as a positive horizontal force is required to produce sidesway. The extension to the buckling problem of a multi-story bent looses some of this physical clarity since the definition of a "positive" force (that is, one operating in the direction of the assumed horizontal motion of the single-story frame) now requires a more abstract statement in terms of the work done by the force system or, equivalently, of the character of the associated stiffness matrix."

Development of Basic Equations

In what follows, the stability of a structure of the type shown in Fig. 3 will be investigated. This structure is subjected to a set of loads, and it is desired to determine the factor of safety λ of the structure against elastic buckling—that is, the parameter λ by which all loads must be multiplied in order to obtain neutral equilibrium.

These loads are usually applied both at the panel points and along the girders. For the purpose of this discussion, however, they will be replaced by a set of statically equivalent panel point loads as shown. Although this represents a deviation from the actual problem, its effect on the solution is small, as was demonstrated by Chwalla [4]. In any case, the effect of primary bending moments on the lateral stability of bents is also ignored in the references [1], [2], and [3] and will be assumed to be negligible in this paper.

In Fig. 1 is shown a typical column in the ith story. If λS_i is the axial force in this column, the lateral deflection $y_i(x)$ is governed by the equation

$$EI\, y_i^{IV} + \lambda S_i\, y_i'' = 0 \qquad (0 \le x \le L_i) \tag{1}$$

in which E and I have the usual meaning and in which primes are employed to denote derivatives. If Y_i denotes the horizontal movement of the ith floor (that is, the floor above the ith story) and if all the quantities are taken as positive as shown in Fig. 1, the joint moments in the column can be expressed as follows:

$$\left.\begin{aligned} M_{i-1}^{i} &= \frac{EI}{L} K\left[(1+C)\frac{Y_i - Y_{i-1}}{L_i} - \theta_{i-1} - C\theta_i\right] \\ M_i^{i-1} &= \frac{EI}{L} K\left[(1+C)\frac{Y_i - Y_{i-1}}{L_i} - \theta_i - C\theta_{i-1}\right] \end{aligned}\right\} \tag{2}$$

In Eqs. (2) the "stiffness factor" K and "carry-over factor" C are given, respectively, by [4]

$$K = \frac{c}{c^2 - s^2} \qquad C = \frac{s}{c} \tag{3}$$

4. For a derivation of these equations see, for instance, [3], p. 211. The notation employed in [3] for K and C are, respectively, C and S/C. K/4 and C (called here S and C) are tabulated in [5] as functions of ϕ.

in which

$$c = \frac{1}{\phi^2}(1 - \phi \cot \phi)$$
$$s = \frac{1}{\phi^2}(\phi \csc \phi - 1) \tag{4a}$$

with ϕ defined by

$$\phi^2 = \frac{\lambda S L^2}{EI} \tag{4b}$$

The equations of equilibrium against rotation of a joint on the ith floor are

$$\sum_j M_i^j = 0 \tag{5}$$

where the summation extends over all the members connected at the joint. As far as the girder moments are concerned, they are determined by the customary equations which are similar to Eqs. (2)—that is, the slope-deflection equations which apply in the absence of axial forces.

It follows further from the condition of equilibrium against horizontal movement that the horizontal force V_i in a column of the ith story is given by (see Fig. 1)[5]

$$V_i = \frac{1}{L_i}\left[M_{i-1}^i + M_i^{i-1} - \lambda S_i(Y_i - Y_{i-1})\right] \tag{6}$$

It follows finally that the horizontal force H_i to be applied at the ith floor is

$$H_i = \sum (V_i - V_{i+1}) \tag{7}$$

in which the summation extends over all the bays.

There are as many equations of equilibrium (5) as there are unknown joint rotations in θ_i It is therefore possible to substitute Eqs. (2) into Eqs. (5) and to solve for the quantities θ_i in terms of the floor translations Y_i.[6] With the joint rotations thus eliminated, substitution of Eqs. (6) into Eq. (7), in consideration of Eqs. (2), leads to the following set of equations:

$$H_i = \sum_{j=1}^{n} a_{ij} Y_j - \lambda \sum_{j=1}^{n} b_{ij} Y_j \quad (i = 1, 2, \cdots n) \tag{8}$$

which may be written in abbreviated matrix notation as follows:

$$H = (A - \lambda B) Y \tag{8'}$$

In Eq. (8) the matrix $A = [a_{ij}]$ is symmetrical and corresponds to that portion of the horizontal force associated with the column moments at the joints.

5. The force V_i is a horizontal force and is not to be confused with the shear in the column.
6. A sufficient condition for the existence and uniqueness of this solution is the requirement that $0 \leq \lambda < \lambda'$, with λ' defined as the multiplier which corresponds to neutral equilibrium without sidesway.

The establishment of A, which is a function of λ, is described below. On the other hand, the matrix $B = [b_{ij}]$, which is also symmetrical, is identified by

$$\left.\begin{aligned} b_{ii} &= \frac{\Sigma S_i}{L_i} + \frac{\Sigma S_{i+1}}{L_{i+1}} \\ b_{i\,i-1} &= -\frac{\Sigma S_i}{L_i} \qquad b_{i\,i+1} = -\frac{\Sigma S_{i+1}}{L_{i+1}} \end{aligned}\right\} \tag{9}$$

while all terms other than the diagonal ones and the ones adjacent to the diagonal terms vanish. The summations in Eqs. (9) extend over all the bays for each story. It is easy to show that B is positive definite.

The matrix A - λ B is positive definite if the equilibrium of the structure is stable. Neutral equilibrium—that is, lateral buckling,—is indicated by the smallest value of $\lambda=\lambda_1$ for which the matrix ceases to be positive definite. This is given by the characteristic equation

$$|A(\lambda_1) - \lambda_1 B| = 0 \tag{10}$$

The fact that the positive definiteness of the matrix A - λB is a necessary condition for the stability of the equilibrium of the structure follows from the principle of Rayleigh, according to which

$$Q(u,u;\lambda) = \Sigma\int EI\,u''^2\,dx - \lambda\Sigma\int S\,u'^2\,dx > 0$$

for any non-trivial set of functions u(x) for stable equilibrium. The functions u(x) are arbitrary except for the restriction that they display a sufficient degree of smoothness to make the integrations meaningful and that they satisfy the geometric boundary conditions, that is, that the continuity of the structure in the joints be preserved. If the set of functions u(x) is further restricted by the requirements that they satisfy Eqs. (1) [u(x) = y(x)] it can readily be shown by a series of partial integrations and in view of the pertinent boundary conditions that

$$Q(y,y;\lambda) = Y\cdot(A-\lambda B)Y$$ [7]

As for the sufficiency of the condition of positive definiteness, it may be assumed in general that

$$u = y + y^*$$

in which y(x) satisfies Eqs. (1) as before and $y^*(x)$ is arbitrary except for the restriction that it vanish at the ends and that the slopes be continuous at the joints. Since

$$Q(y+y^*, y+y^*; \lambda) = Q(y,y;\lambda) + 2Q(y,y^*;\lambda) + Q(y^*,y^*;\lambda)$$

7. This notation is employed to denote the inner product of two vectors.

and since the bi-linear term can readily be shown to vanish, the sufficiency condition applies if the last term in the equation shown above is positive definite. This is the case provided $\lambda < \lambda'$, in which λ' is defined as before. In this manner, the sufficiency condition of the positive definiteness of the matrix $A - \lambda B$ is seen to apply for the range which is of interest for the solution of the problem under consideration.

The direct solution of the characteristic Eq. (10) presents almost insurmountable numerical difficulties since the parameter λ appears both explicitly and implicitly in the form of rather involved transcendental functions. It is possible, however, to establish upper and lower bounds to the root of Eq. (10) by means of the following theorem:

Let $\Lambda(\lambda)$ be the smallest root of the algebraic equation

$$|A(\lambda) - \Lambda(\lambda)B| = 0 \tag{11}$$

Then λ_1 is bounded between Λ and λ.

This theorem, which is proved in the appendix, makes it possible to establish upper and lower bounds to the smallest safety factor λ_1 through a relatively simple numerical procedure. As will be shown in the next section, the matrix A can be computed, for an assumed value of λ, by means of a series of moment distribution processes, while the matrix B is a constant. If now, for such an assumed value of λ, Eq. (11) is solved for Λ, the theorem states that the unknown buckling force lies between the initially assumed value and that obtained from the solution of Eq. (11). With upper and lower bounds thus established, the next step is to narrow the gap to within the desired degree of accuracy, as is described in more detail below.

Establishment of the Matrix $A = [a_{ij}]$

In order to establish the coefficients of the matrix A, it is well to remember that these coefficients ($a_{ij} = a_{ji}$) represent the force to be applied at the jth floor in order to obtain a unit displacement at the ith floor, all floors other than the ith being held against sidesway; in the computation of these forces, the direct effect of the axial force on the shears in the columns is to be ignored.

The procedure in establishing the set of values a_{ij} is now straight forward. If a value of λ is assumed to start with, all the coefficients K and C can be computed from Eqs. (3) or taken from [5]. If now the ith floor is subjected to a unit movement, while all other floors are prevented from translation, it is possible to determine all fixed-end-moments by means of the Eqs. (2), with the rotations θi, of course, set equal to zero. Naturally, these fixed-end-moments will occur only at the ith floor and at the floors immediately above and below the ith floors; the sign conventions are those commonly employed in structural design, that is moments acting clockwise on a joint are considered positive.

With the fixed-end-moments thus determined, all joints are released and permitted to rotate until equilibrium is reached. Again, standard structural design techniques can be employed during the moment distribution process, subject, of course, to the modification of the distribution and carry-over factors corresponding to the presence of axial forces.[8] It may be noted that, according to [7], the convergence of the moment distribution process is assured provided that $\lambda < \lambda'$. In other words, the assumed value of the axial

8. This effect is discussed in Ref. [6].

forces must be such as to correspond to a stable structure if sidesway were prevented; this constitutes no restriction from a practical point of view, as was discussed above.

After all unbalanced moments are thus removed from the joints, the shears in the columns are computed by conventional means (as the sum of the two end moments divided by the length of the column) and the forces at the jth floor are finally determined as the difference between the sum of the column shears below and above that floor. The set of forces thus obtained constitute the ith row in the matrix A.

The same process is now repeated by giving unit translations to all the other floors and by computing the forces corresponding to these unit sidesways. It may be mentioned parenthetically that the method is somewhat self-checking in that the matrix A is symmetrical; this can be seen from the fact that $A - \lambda B$ is symmetrical by Maxwell's reciprocity theorem and that B is itself symmetrical. An error in the moment distribution process is therefore likely to show up in a lack of symmetry in the resulting matrix A.

The numerical solution of the determinantal equation (11) can be obtained in a number of ways and will not be discussed here. For a small number of floors ($n = 3$ or less), the solution can be obtained directly without much difficulty. For taller structures, an iterative approach is likely to be more convenient.

After the first set of lower and upper bounds has thus been determined, the gap can be narrowed down fairly rapidly. In making a second approximation, the next trial value of λ should generally be chosen much closer to the value of Λ obtained from Eq. (11) than to the value of λ assumed initially. This can be seen from the fact that the derivative of Λ with respect to λ, while always negative (see Appendix) is usually much smaller than unity in absolute value. For $\lambda = 0$, the slope of the $\Lambda - \lambda$ curve lies between about -5% and -20% depending on the ratio of the floor stiffnesses to the column stiffnesses. So rapid, in fact, is the process that it should rarely be necessary to apply it more than twice, even if a comparatively incorrect initial assumption is made. In the examples described below, the initially assumed value of λ is zero, which is an obvious absurdity; nevertheless, the relative discrepancy between the upper and lower bound, after the second trial, is about two per cent, which is well within the degree of accuracy desired for a problem of this type.

Numerical Examples

In this section, the procedure outlined above is illustrated by means of two numerical examples represented by Figs. 2 and 3. Both examples deal with relatively simple, although not necessarily unrepresentative structures, which results in a considerable reduction in the numerical labor invo... 1.. On the other hand, it must be realized that frames which display less symmetry or a greater number of stories, while requiring a larger numerical effort in the use of the method described here, are virtually beyond exact analysis as the establishment and solution of the basic determinantal equation becomes prohibitively difficult.

The two-story bent shown in Fig. 2 is analyzed completely in Ref. [3], p. 255 ff. and is introduced here for purposes of comparison and verification. It is assumed that the beam stiffness I_i/L and column stiffness I/h are equal ($\gamma = 1$ in [3]) and that the only forces acting on the structures are those applied at the second floor as shown. With these assumptions, an initial (and obviously incorrect) set of two moment distribution procedures is

carried out for the case of $\lambda = 0$, which is especially simple since all the stiffness and carry-over coefficients are the same as the ones which are used in conventional structural design. Using only slide rule accuracy this leads to the determinantal equation (corresponding to Eq. (11))

$$\begin{vmatrix} 40.60 - 4\Lambda & -17.65 + 2\Lambda \\ -17.65 + 2\Lambda & 13.42 - 2\Lambda \end{vmatrix} = 0$$

whose smaller root is given by

$$\Lambda(0) = 5.54$$

Similarly, a second attempt, with the assumed value of $\lambda = 5.00$, results in the equation

$$\begin{vmatrix} 37.10 - 4\Lambda & -16.35 + 2\Lambda \\ -16.35 + 2\Lambda & 12.56 - 2\Lambda \end{vmatrix} = 0$$

with

$$\Lambda(5.00) = 5.19$$

as its smaller root. The exact value of the safety factor has thus been boxed in between 5.00 and 5.19. It is seen that the average slope of the $(\Lambda - \lambda)$ curve is -0.07 between $\lambda = 0$ and $\lambda = 5.00$. By extrapolation, an approximate solution to the equation

$$\Lambda(\lambda_1) = \lambda_1$$

is $\lambda_1 = 5.17$. On the other hand, it is seen that Eq. (488) in Ref. [3], for the case considered, is satisfied by $\lambda_1 = 5.15$.

The second example used to illustrate the method outlined in this paper is shown in Fig. 3. Starting again with an assumed value of $\lambda = 0$, the first trial yields the smallest root $\Lambda = 4.75$. On the basis of an assumed slope of about -10% at $\lambda = 0$, the second set of calculations is carried out for $\lambda = 4.3$, which results in the lowest root $\Lambda = 4.4$. Within the degree of accuracy employed, which is sufficient in most practical cases, there is therefore no need to carry the calculations any further.

CONCLUSION

A numerical method has been described which permits the determination of the factor of safety against buckling by sidesway through a sequence of steps leading to the establishment of upper and lower bounds to the desired buckling force. The methods involved are mainly those common in routine design operations and do not involve the solution of complicated transcendental equations.

In the design of columns, present building codes do not take into consideration the end conditions of the columns, but treat all cases as being simply supported. On the other hand, it is apparent that the buckling forces in buildings which are not held against sidesway may be considerably less than those corresponding to instability of simply supported columns. An additional

reason for a more careful analysis of the problem is to be found in the fact that the buckling mode of the type of structure under consideration is very similar to its response to lateral forces such as wind, earthquake forces, and the like. Accordingly, the presence of axial forces may have a much greater effect on the magnitude of this response than is customarily realized.

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APPENDIX

In order to prove the theorem relating to Eq. (11) it is considered that the structure is subjected to a given set of horizontal floor forces identified by the non-vanishing vector H. If now the deflection functions $y_i(x)$ are considered differentiable functions of the parameter λ, the following equation is obtained by differentiating Eq. (1) with respect to λ:

$$EI\dot{y}_i^{IV} + \lambda S_i \dot{y}_i'' = -S_i y_i'' \qquad (12)$$

In Eq. (12) and in what follows, a dot above a letter identifies differentiation with respect to λ.

In view of Eqs. (1) and (12), it is possible to establish the relationship

$$\sum \int S y'^2 dx = H \cdot \dot{Y} \qquad (13)$$

in which the summation extends over all the columns. Eq. (13) is obtained by a number of partial integrations and in view of the boundary conditions relating to the deflection functions y(x) and their derivatives $\dot{y}(x)$. On the other hand, by differentiating Eq. (8) with respect to λ, and in view of Eq. (8) and of the symmetry of the matrices involved, the right side of Eq. (13) can be expressed as follows:

$$H \cdot \dot{Y} = Y \cdot (B - \dot{A})Y \qquad (14)$$

Thus, by combining Eqs. (13) and (14), the relationship

$$Y \cdot \dot{A} Y = Y \cdot BY - \sum \int S y'^2 dx < 0 \qquad (15)$$

is obtained.

The fact that the right side of Eq. (15) is negative can readily be verified. For if, in general,

$$y_i(x) = Y_{i-1} + \frac{Y_i - Y_{i-1}}{L_i} x + \eta_i(x)$$

in which $\eta_i(o)$ and $\eta_i(L_i)$ vanish, the right side of Eq. (15) can be reduced to

the form

$$-\sum \int S \eta'^2 dx$$

This expression is never positive since all S_i are positive (compression); its vanishing implies that all columns under axial compression remain straight, which is ruled out.

Since the displacement vector Y is arbitrary, it follows from (15) that A is negative definite. This in turn implies that Λ is a monotonically decreasing function of λ. This can be seen by differentiating the identity

$$A(\lambda) Y^{(1)}(\lambda) \equiv \Lambda(\lambda) B Y^{(1)}(\lambda) \tag{16}$$

with respect to λ and by dot-multiplying both sides by $Y^{(1)}$. In view of Eq. (16) and of the symmetry of A and B, this leads to the result

$$\dot{\Lambda} = \frac{Y^{(1)} \cdot \dot{A} Y^{(1)}}{Y^{(1)} \cdot B Y^{(1)}} < 0 \tag{17}$$

The negativeness of the fraction in Eq. (17) follows from Eq. (15) and from the positive definiteness of B.

Λ is therefore monotonically decreasing in λ. Since furthermore

$$\Lambda(\lambda_1) = \lambda_1 \tag{18}$$

it follows that

$$\Lambda(\lambda) \gtreqless \lambda_1 \quad \textit{according as} \quad \lambda \lesseqgtr \lambda_1 \tag{19}$$

This completes the proof of the theorem.

It is of interest to note, however, that the theorem applies only in the region in which Λ is regular—that is, for $0 \leq \lambda < \lambda'$. λ' is defined as the smallest root of the equation

$$|A^{-1}(\lambda')| = 0 \tag{20}$$

It is seen that the definition of λ' by means of Eq. (20) coincides with the one used previously, since the singularity of A^{-1} implies a mode of buckling without sidesway. Since λ_1 is always less than λ', and in fact usually by a considerable proportion, the restriction placed on the validity of the theorem is therefore of no practical significance. It may also be of interest to note that since A has a pole at $\lambda = \lambda'$, it may become positive definite again for values of $\lambda > \lambda'$. This explains why the sufficiency condition for stable equilibrium applies only in the restricted range mentioned above.

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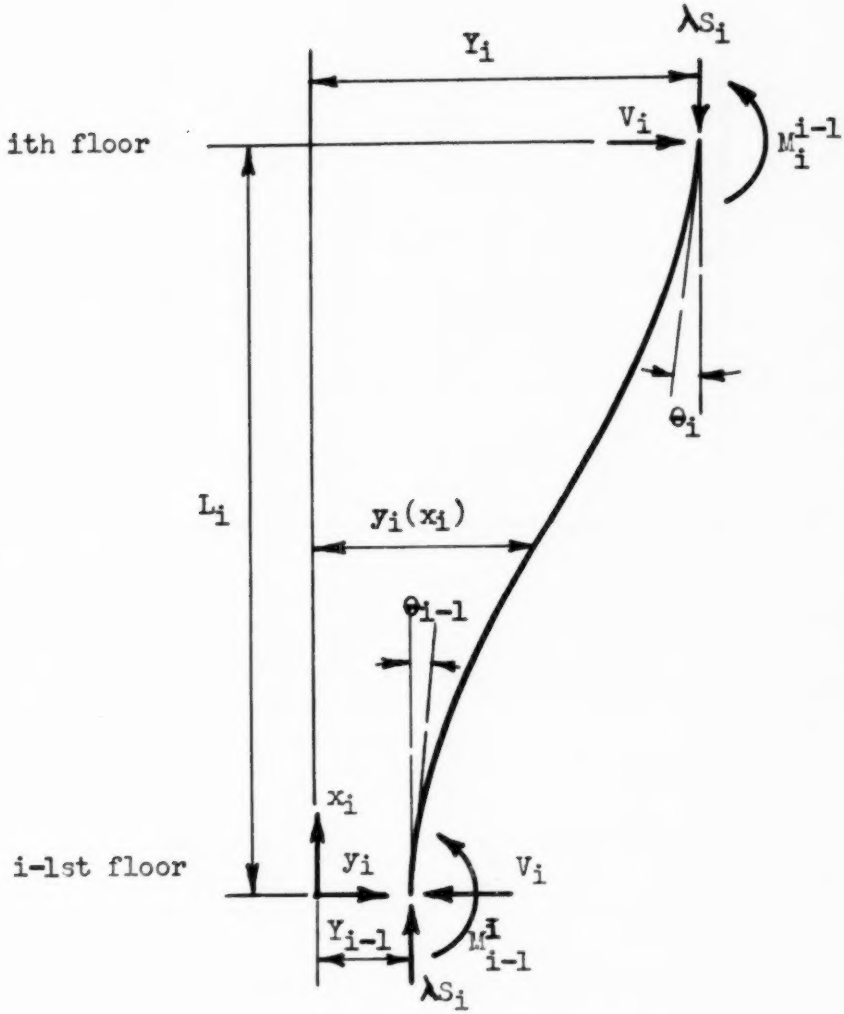

λS_i
Y_i
V_i
ith floor
M_i^{i-1}
θ_i
L_i
y_i(x_i)
θ_{i-1}
x_i
i-1st floor
y_i
V_i
Y_{i-1}
M_{i-1}^i
λS_i

Fig. 1

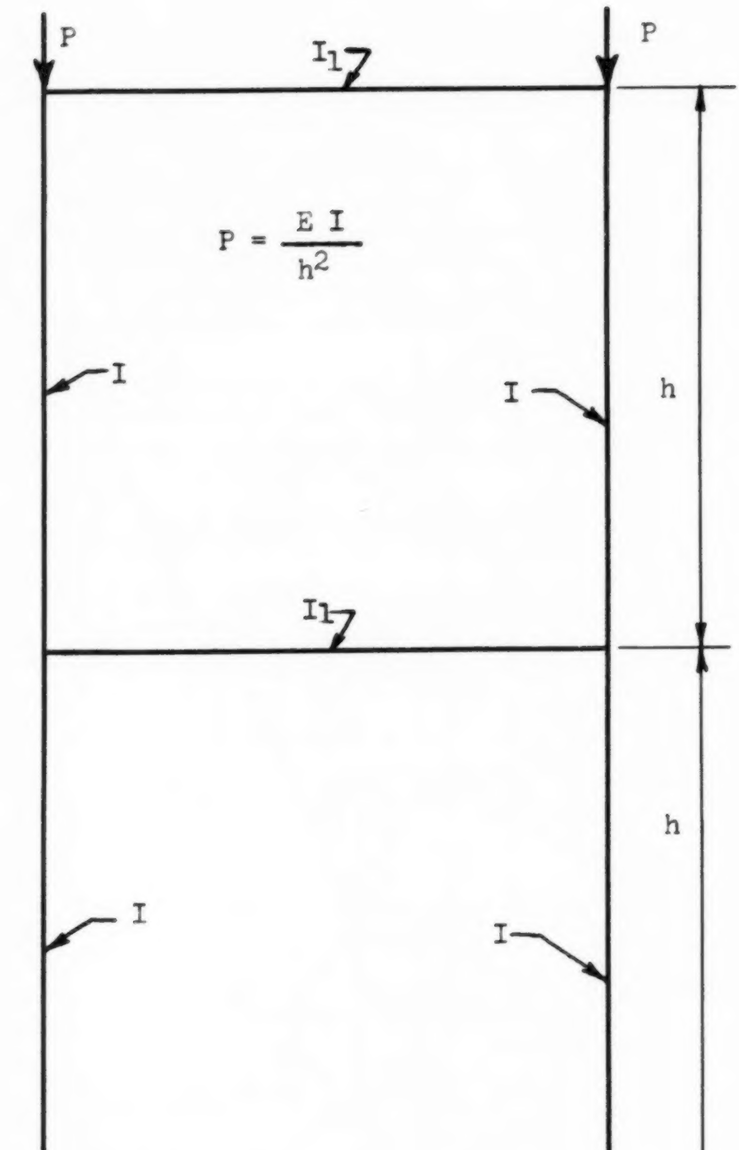

Fig. 2

$$P = \frac{EI}{L^2}$$

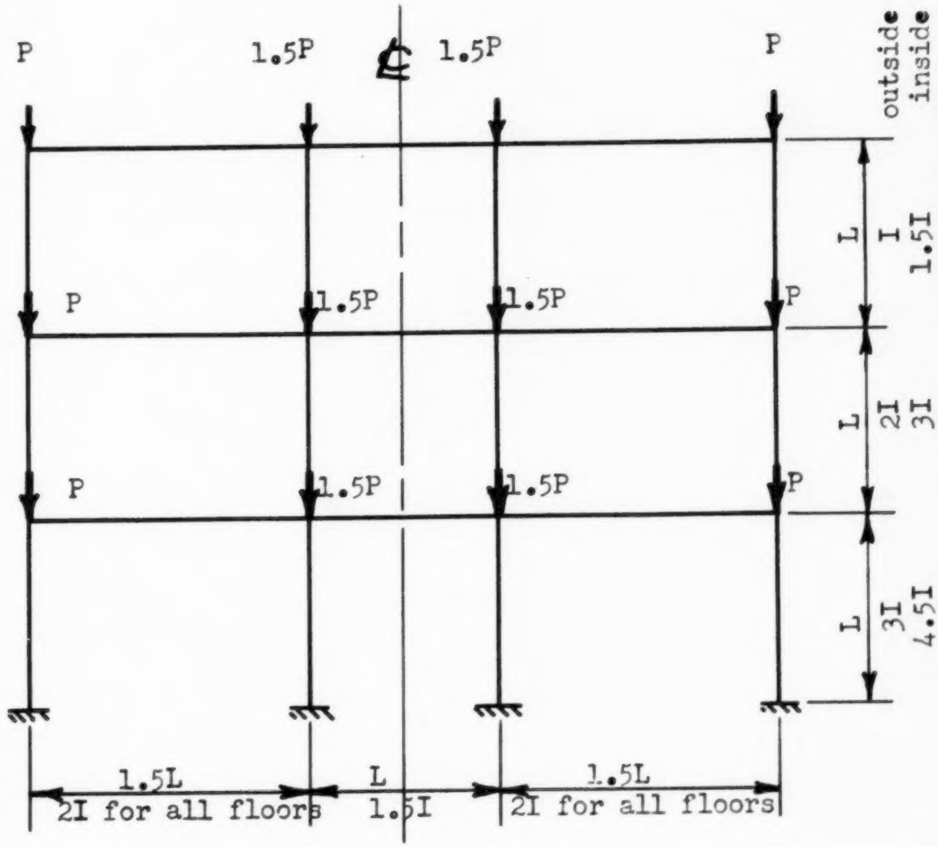

Fig. 3